# Cedar Crest College

## Calculus IV

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# Lecture Notes

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# Preface

This is meant as a teaching aid. It can be freely distributed and edited in any way. For a copy of the LaTeX document, please email the author. These notes are adapted from James Stewart's *Calculus: Early Transcendentals* Eighth Edition.

# Contents

# Chapter 12

# Vectors and the Geometry of Space

## 12.1 Operations in 3-Space

We have a lot to cover this semester; however, it is important to have a good foundation before we trudge forward. In that vein, let's review vectors and their geometry in space ($\mathbb{R}^3$) briefly.

### 12.1.1 Points

**Definition 1** (Distance)**.** Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in $\mathbb{R}^3$. Then the **distance** from $P$ to $Q$, denoted $d(P, Q)$ is

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

**Problem 2.** Show that $(x - y)^2 = (y - x)^2$.

**Definition 3** (Planes in $\mathbb{R}^3$)**. Planes** in $\mathbb{R}^3$ are denoted as $x = a, y = b$, or $z = c$, where $a, b$, and $c$ are real numbers (henceforth we will denote the fact that $a, b$, and $c$ are real numbers as $a, b, c \in \mathbb{R}$.)

**Recall 4.** A **circle** in $\mathbb{R}^2$ is defined to be all of the points in the plane ($\mathbb{R}^2$) that are equidistant from a central point.

A natural generalization of this to 3-space would be to say that a **sphere** is defined to be all of the points in $\mathbb{R}^3$ that are equidistant from a central point $C$. This is exactly what the following definition does!

**Definition 5** (Sphere)**.** Let $C = (h, k, l)$ be the center of a sphere. Then the **sphere** centered at $C$ with radius $r$ is defined by the equation

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2.$$

That is to say that this defines all points $(x, y, z) \in \mathbb{R}^3$ that are a distance $r$ from the center point of the sphere.

### 12.1.2 Vectors

**Definition 6** (Vector)**.** A **vector** is a mathematical object that stores both length (which we will call magnitude) and direction.

**Definition 7** (Vector Between Two Points)**.** Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ . Then the **vector** with initial point $P$ and terminal point $Q$ (denoted $\overrightarrow{\mathbf{PQ}}$) is defined by

$$\overrightarrow{\mathbf{PQ}} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

**Definition 8** (Vector Equality)**.** Two vectors are said to be **equal** if and only if they have the same length and direction, regardless of their position in $\mathbb{R}^3$. That is to say that a vector can be moved anywhere in space as long as the magnitude and direction are preserved.

**Convention 9.** For convenience, we use something called the position vector to denote the family of vectors with a given direction and magnitude. The **position vector**, $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ , is formed by making the initial point of the vector the origin, $O = (0, 0, 0)$ , and terminal point $(v_1, v_2, v_3)$

**Definition 10** (Magnitude (A.K.A. Length or Norm))**.** Let $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ . Then the **magnitude** of $\vec{\mathbf{v}}$ (denoted $|\vec{\mathbf{v}}|$ or sometimes $\|\vec{\mathbf{v}}\|$) is defined by

$$|\vec{\mathbf{v}}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

### 12.1.3 Operations

**Definition 11** (Vector Addition)**.** Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle .$$

**Definition 12** (Scalar Multiplication)**.** Let $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ and $k \in \mathbb{R}$. Then

$$k\vec{\mathbf{v}} = \langle kv_1, kv_2, kv_3 \rangle .$$

**Definition 13** (Unit Vector)**.** A **unit vector** is a vector whose magnitude is 1. Note that we can given a vector $\vec{\mathbf{v}}$, we can form a unit vector $\widehat{\mathbf{v}}$ by dividing by the magnitude of $\vec{\mathbf{v}}$. That is to say, Let $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ . Then

$$\widehat{\mathbf{v}} = \frac{1}{|\vec{\mathbf{v}}|} \langle v_1, v_2, v_3 \rangle .$$

**Definition 14** (Standard Vectors)**.** Any vector can be denoted as the linear combination of the **standard unit vectors** $\widehat{\mathbf{i}} = \langle 1, 0, 0 \rangle$ , $\widehat{\mathbf{j}} = \langle 0, 1, 0 \rangle$ , and $\widehat{\mathbf{k}} = \langle 0, 0, 1 \rangle$ . So given a vector $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ , one can express it with respect to the standard vectors as

$$\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle = v_1\widehat{\mathbf{i}} + v_2\widehat{\mathbf{j}} + v_3\widehat{\mathbf{k}}.$$

This text, however, will more often than not use the angle brace notation.

**Definition 15** (Dot Product)**.** Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then the dot product or Euclidean Inner Product as it is sometimes referred is

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1v_1 + u_2v_2 + u_3v_3 = |\vec{\mathbf{u}}| \, |\vec{\mathbf{v}}| \cos(\theta) .$$

**Theorem 16.** Two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are **orthogonal** if and only if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

**Problem 17.** Show that if two non-zero vector are orthogonal then $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

**Definition 18** (Cross Product)**.** Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then the cross product is the determinant of the following matrix:

$$\begin{aligned}
\vec{\mathbf{u}} \times \vec{\mathbf{v}} &= \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \widehat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \widehat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \widehat{\mathbf{k}} \\
&= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle .
\end{aligned}$$

**Observation 19.** The cross product of two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ gives us a vector that is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

**Definition 20** (Equivalent to Cross Product)**.** Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then the cross product can also be defined as

$$|\vec{\mathbf{u}} \times \vec{\mathbf{v}}| = |\vec{\mathbf{u}}| \, |\vec{\mathbf{v}}| \sin(\theta) .$$

**Problem 21.** Show that is two non-zero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel if and only if $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$.

## 12.2 Equations in 3-Space

**Equation 22** (Parametrization of a Line)**.** Let $O = (0,0,0)$ be the origin in $\mathbb{R}^3, P_0 = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$, and $\vec{\mathbf{v}} = \langle A, B, C\rangle$ be a vector in $\mathbb{R}^3$ parallel to the line being parametrized. Then the line through $P_0$ parallel to $\vec{\mathbf{v}}$ is

$$\vec{\mathbf{r}}(t) = \overrightarrow{\mathbf{OP_0}} + t\vec{\mathbf{v}} \qquad t \in \mathbb{R}.$$

This can also be written as

$$x = x_0 + At, \qquad y = y_0 + Bt, \qquad z = z_0 + Ct \qquad t \in \mathbb{R}.$$

or as the symmetric equation

$$\frac{x - x_0}{A} = \frac{y - y_o}{B} = \frac{z - z_0}{C}.$$

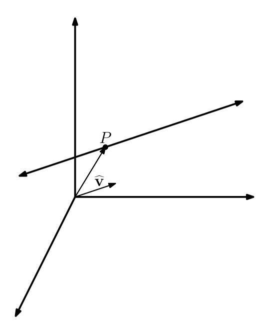


Figure 12.1: Parametrization of a Line

**Equation 23** (Parametrization of a Line Segment)**.** Let $O$ denote the origin, $P$ be the initial point of a line segment, and $Q$ be the terminal point of a line segment. Then the line segment $\overline{PQ}$ can be parametrized as

$$\vec{\mathbf{r}}(t) = (1-t)\overrightarrow{\mathbf{OP}} + t\overrightarrow{\mathbf{OQ}} \qquad 0 \le t \le 1.$$

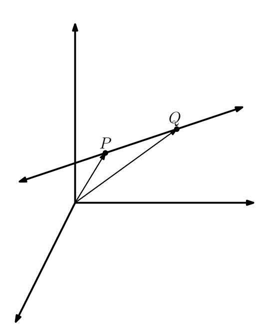


Figure 12.2: Parametrization of a Line Segment

**Problem 24.** Find a vector equation and parametric equation for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\langle 1, 4, -2\rangle$.

**Problem 25.** Find the parametric Equation of the line segment from $(2, 4, -3)$ to $(3, -1, 1)$.

**Equation 26** (Planes)**.** Let $P_0 = (x_0, y_0, z_0)$ be a point in the plane and $\vec{\mathbf{n}} = \langle a, b, c\rangle$ be a vector normal to the plane. Then the equation of the plane is

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0.$$

## 12.3 Cylinders and Quadric Surfaces

**Definition 27** (Cylinder)**.** A **cylinder** is a surface that consists of all lines that are parallel to a given line and pass through a given plane curve.

**Problem 28.** Sketch $y = x^2$ in $\mathbb{R}^3$.

**Problem 29.** Sketch $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

**Problem 30.** Sketch $y^2 + z^2 = 1$ in $\mathbb{R}^3$.

**Definition 31** (Quadric Surfaces). A **quadric surface** is the graph of a second-degree equation in three variables $x, y$, and $z$. By translation and rotation, we can write the standard form of a quadric surface as

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0.$$

**Definition 32** (Trace). The **trace** of a surface in $\mathbb{R}^3$ is the graph in $\mathbb{R}^2$ obtained by allowing one of the variables to be a specific real number. For example, $x = a$.

**Problem 33.** Use the traces of the quadric surface to sketch $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$.

**Problem 34.** Use the traces to sketch $z = 4x^2 + y^2$.

**Problem 35.** Sketch $z = y^2 - x^2$.

**Problem 36.** Sketch $\dfrac{x^2}{4} + y^2 - \dfrac{z^2}{4} = 1$

Suggested Problems: Section $12.6 : 3 - 8, 11 - 20$

# Chapter 13

# Vector Functions

## 13.1 Vector Functions and Space Curves

**Definition 37** (Vector Function (or Vector Valued Function))**.** A **vector function** is a function whose domain is $\mathbb{R}$ and whose range is a set of vectors. That is, a vector function can be written as $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$ .

**Problem 38.** What is the domain of $\vec{\mathbf{r}}(t) = \left\langle t^3, \ln{(3-t)}\, , \sqrt{t} \right\rangle$

**Theorem 39.** If $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$ , then $\lim\limits_{t \to a} \left\langle \lim\limits_{t \to a} f(t), \lim\limits_{t \to a} g(t), \lim\limits_{t \to a} h(t) \right\rangle$ .

**Problem 40.** Find $\lim\limits_{t \to 0} \vec{\mathbf{r}}(t)$ given $\vec{\mathbf{r}}(t) = \left\langle 1 + t^3, te^{-t}, \dfrac{\sin{(t)}}{t} \right\rangle$

**Definition 41** (Continuity at a Point)**.** A vector function $\vec{\mathbf{r}}(t)$ is **continuous at a point** $a$ if $\lim_{t\to a} \vec{\mathbf{r}}(t) = \vec{\mathbf{r}}a$. That is to say that if $f(t), g(t)$, and $h(t)$ are continuous at $a$, then $\vec{\mathbf{r}}(t)$ is continuous at $a$.

**Definition 42** (Space Curve)**.** If $\vec{\mathbf{r}}(t)$ is continuous on an interval $I$, then $\vec{\mathbf{r}}$ is called a **space curve** over the interval $I$.

**Definition 43.** A space curve $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle$ has parametrization

$$x = f(t) \qquad y = g(t) \qquad z = h(t) \qquad t \in \operatorname{dom}\left(\vec{\mathbf{r}}(t)\right).$$

**Problem 44.** Describe and parametrize $\vec{\mathbf{r}}(t) = \langle 1+t, 2+5t, -1+6t\rangle$.

**Problem 45.** Sketch $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t\rangle$.

**Problem 46.** Find a vector equation and a parametric equation for the line segment that joins the point $P = (1, 3, -2)$ to $Q = (2, -1, 3)$.

**Problem 47.** Find the vector function that represents the curve of the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Suggested Problems: Section $13.1 : 7 - 14, 28, 30$.

# 13.2 Derivatives and Integrals of Vector Functions

## 13.2.1 Derivatives

**Definition 48** (Vector Function Derivative)**.** If $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$ be a vector function where $f, g$, and $h$ are differentiable functions, then

$$\vec{\mathbf{r}}'(t) = \langle f'(t), g'(t), h'(t) \rangle .$$

**Problem 49.** Find the derivative of $\vec{\mathbf{r}}(t) = \langle 1 + t^3, te^{-t}, \sin(2t) \rangle$ and use it to find the unit tangent vector at $t = 0$.

**Problem 50.** Find the parametric equation for the tangent line to the helix $x = 2\cos(t), y = \sin(t), z = t$ at the point $\left(0, 1, \frac{\pi}{2}\right)$.

**Theorem 51.** When taking derivative, all of the standard methods work, but now we have two different types of multiplication. Hence, the following two operations hold:

- $\dfrac{d}{dt}\left(\vec{\mathbf{u}}\,(t)\cdot\vec{\mathbf{v}}\,(t)\right)=\vec{\mathbf{u}}'(t)\cdot\vec{\mathbf{v}}(t)+\vec{\mathbf{u}}(t)\cdot\vec{\mathbf{v}}'(t)$.
- $\dfrac{d}{dt}\left(\vec{\mathbf{u}}\,(t)\times\vec{\mathbf{v}}\,(t)\right)=\vec{\mathbf{u}}'(t)\times\vec{\mathbf{v}}(t)+\vec{\mathbf{u}}(t)\times\vec{\mathbf{v}}'(t)$.

### 13.2.2 Integrals

**Definition 52** (Vector Function Integration)**.** Let $\vec{\mathbf{r}}(t)=\langle f(t),g(t),h(t)\rangle$ be a vector function. Then $\displaystyle\int_a^b\vec{\mathbf{r}}(t)\,dt=\left\langle\int_a^b f(t)\,dt,\int_a^b g(t)\,dt,\int_a^b h(t)\,dt\right\rangle$

**Problem 53.** Evaluate $\int_0^{\frac{\pi}{2}}\vec{\mathbf{r}}(t)\,dt$ given $\vec{\mathbf{r}}(t)=\langle 2\cos(t),\sin(t),2t\rangle$ .

Suggested Problems: Section $13.2:9-26,35-40$.

# 13.3 Arc Length and Curvature

## 13.3.1 Arc Length

**Definition 54** (Arc Length)**.** Let $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle$ be a vector function. Then the **arc length** (sometimes just referred to as **length**) is

$$\begin{aligned} L &= \int_a^b \left|\lim_{h\to 0} \tfrac{\vec{\mathbf{r}}(t+h)-\vec{\mathbf{r}}(t)}{h}\right| \, dt \\ &= \int_a^b \left|\vec{\mathbf{r}}'(t)\right| \, dt \\ &= \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2 + \left[h'(t)\right]^2} \, dt \\ &= \int_a^b \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dz}{dt}\right)^2} \, dt. \end{aligned}$$

**Notation 55.** $\dfrac{ds}{dt} = \left|\vec{\mathbf{r}}'(t)\right|.$

**Definition 56** (Arc Length Function)**.** Let $s(t)$ represent the length of a curve from $\vec{\mathbf{r}}(a)$ to $\vec{\mathbf{r}}(t)$. Then $s(t)$ is defined as

$$s(t) = \int_a^t \left|\vec{\mathbf{r}}'(u)\right| \, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} \, du.$$

**Problem 57.** Find the length of the arc of the circular helix with vector equation $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t\rangle$ from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

### 13.3.2 Curvature

**Definition 58** (Curvature)**.** The **curvature**, $\kappa$, of a curve $C$ at a given point is a measure of how quickly the curve changes directions at that point. That is to say that it is the magnitude of the rate of change of the unit tangent vector with respect to the arc length. Hence, we have

$$\kappa = \left|\frac{d\vec{\mathbf{T}}}{ds}\right| = \frac{\left|\vec{\mathbf{T}}'(t)\right|}{|\vec{\mathbf{r}}'(t)|}, \qquad \text{where} \qquad \vec{\mathbf{T}} = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}.$$

A sometimes more convenient way to use this is

$$\kappa = \frac{|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)|}{|\vec{\mathbf{r}}'(t)|^3}.$$

**Problem 59.** Find the curvature of the twisted cubic $\vec{\mathbf{r}}(t) = \langle t, t^2, t^3 \rangle$ at a general point, and at $(0, 0, 0)$.

### 13.3.3 Normal and Bi-normal Vectors

**Definition 60** (Unit Tangent Vector)**.** Given a curve $C$, the **unit tangent vector**, $\vec{\mathbf{T}}$, is the vector that touches the curve at a given point and points in the same direction as the curve.

**Definition 61** ((Unit) Normal Vector)**.** Given a curve $C$, the **normal vector**, $\vec{\mathbf{N}}$, is the derivative of the tangent vector divided by the magnitude. It points *into* the curve. That is to say

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{T}}'(t)}{\left|\vec{\mathbf{T}}'(t)\right|}.$$

**Definition 62** (Bi-normal Vector)**.** The **bi-normal vector**, $\vec{\mathbf{B}}$, is the (unit) vector that is perpendicular to both $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$. That is to say that it is the cross product of the tangent vector and the normal vector. In terms of direction, it obeys the *right hand rule*. In other words,

$$\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}.$$

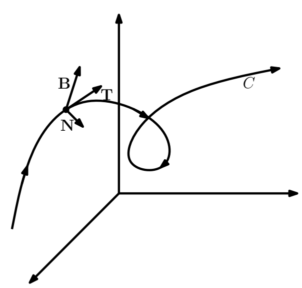


Figure 13.1: Frenet-Serret Equations

**Problem 63.** Find the Frenet-Serret Equations of the circular helix $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$.

**Definition 64** (Normal Plane)**.** The **Normal Plane** is defined by $\vec{\mathbf{N}}$ and $\vec{\mathbf{B}}$. It represents all vectors that are perpendicular to $\vec{\mathbf{T}}$.

**Definition 65** (Osculating (or Kissing) Plane)**.** The **osculating plane** is the plane defined by $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$. It represents the plane that most closely fits the curve at that point.

**Definition 66** (Osculating Circle)**.** The **osculating circle** is the circle in the osculating plane that has the same tangent as the curve $C$ at the point $P$ and lies on the concave side of $C$ with radius $\rho = \frac{1}{\kappa}$.

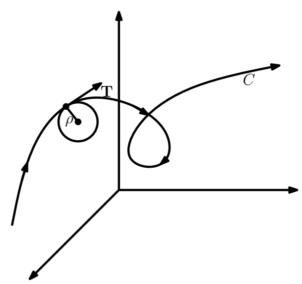


Figure 13.2: Osculating Circle

**Problem 67.** Find the equation of the normal plane and the osculating plane of the helix $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$ at the point $P = \left(0, 1, \frac{\pi}{2}\right)$.

Suggested Exercises: Section 13.3 : $1 - 6, 17 - 25, 49, 50$.

## 13.4 Motion in Space, Velocity, and Acceleration

**Definition 68** (Velocity, Acceleration, & Speed)**.** Let $\vec{\mathbf{r}}(t)$ be a position vector for a particle at time $t$. Then the **velocity** of the particle at time $t$ is $\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t)$ and the **acceleration** of the particle at time $t$ is $\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \vec{\mathbf{r}}''(t)$. The speed of the particle at time $t$ is defined by $|\vec{\mathbf{v}}(t)| = |\vec{\mathbf{r}}'(t)|$.

**Problem 69.** Given a position vector $\vec{\mathbf{r}}(t) = \langle t^3, t^2 \rangle$, find the velocity, speed, and acceleration at $t = 1$.

**Problem 70.** A moving particle starts at an initial position $\vec{\mathbf{r}}(0) = \langle 1, 0, 0 \rangle$, has initial velocity $\vec{\mathbf{v}}(0) = \langle 1, -1, 1 \rangle$, and acceleration is defined by $\vec{\mathbf{a}}(t) = \langle 4t, 6t, 1 \rangle$. Find the particle's position and velocity at time $t$.

# Chapter 14

# Partial Derivatives

## 14.1 Functions of Several Variables

### 14.1.1 Domain and Range

**Definition 71** (Function of Two Variables)**.** A **function of two variables**, $f$, is a rule that assigns each ordered pair of real numbers $(x, y)$ in a set $D \subset \mathbb{R}^2$ a unique real number denoted $z = f(x, y)$. The set $D$ is called the **domain** of $f$ and the set of values that are returned (the $z$ values) are called the **range**. In shorthand, we say

$$f : \mathbb{R}^2 \to \mathbb{R}.$$

**Definition 72.** Let $f$ be a function of two variables, $z = f(x, y)$. Then $x$ and $y$ are called **independent variables** and $z$ is called a **dependent variable**.

**Problem 73.** Let $f(x, y) = \dfrac{\sqrt{x+y+1}}{x-1}$ Evaluate $f(3, 2)$ and give its domain.

**Problem 74.** Find the domain of $f(x, y) = x \ln\left(y^2 - x\right)$.

**Problem 75.** Find the domain and range of $f(x, y) = \sqrt{9 - x^2 - y^2}$.

### 14.1.2 Graphs

**Definition 76** (Graph)**.** If $f$ is a function of two variables with domain $D$, then the **graph** of $f$ is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ for all $(x, y) \in D$.

**Problem 77.** Sketch the graph of $f(x, y) = 6 - 3x - 2y$.

**Problem 78.** Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

### 14.1.3 Level Curves

**Definition 79** (Level Curves)**.** The **level curves** of a function of two variables, $f$, are the curves of the equation $f(x, y) = k$ for some $k \in K \subset \mathbb{R}$.

**Problem 80.** Sketch the level curves of $f(x, y) = 6 - 3x - 2y$ for $k \in \{-6, 0, 6, 12\}$.

**Problem 81.** Sketch the level curves of $g(x, y) = \sqrt{9 - x^2 - y^2}$ for $k \in \{0, 1, 2, 3\}$

**Problem 82.** Sketch the level curves of $h(x, y) = 4x^2 + y^2 + 1$.

Suggested Problems: Section 14.1 : $43 - 50, 65 - 68$.

## 14.2 Limits and Continuity

**Theorem 83** (Limits Along a Path)**.** If $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$ as $(x,y)$ approaches $(a,b)$ along the path $C_1$ and $\lim_{(x,y)\to(a,b)} f(x,y) = L_2 \neq L_1$ as $(x,y)$ approaches $(a,b)$ along the path $C_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

**Problem 84.** Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

**Problem 85.** Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

**Problem 86.** Does $\lim_{(x,y)\to(1,1)} \dfrac{2xy}{x^2+y^2}$ exist?

**Problem 87.** Does $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$ exist?

**Problem 88.** Does $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ exist?

**Definition 89** (Continuity)**.** A function of two variables $f$ is called **continuous** at a point $(a, b) \in \mathbb{R}^2$ if $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

**Problem 90.** Is $\frac{2xy}{x^2+y^2}$ continuous at $(1, 1)$? What about $(0, 0)$? Why?

**Problem 91.** Use the squeeze theorem to show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

**Problem 92.** Where is $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$ continuous?

Suggested Problems: Section $14.2 : 5 - 22, 29 - 38$.

# 14.3 Partial Derivatives

## 14.3.1 First Order Partial Derivatives

**Definition 93** (Partial Derivatives)**.** If $f$ is a function of two variables, its **partial derivatives** are the functions $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ defined by:

$$\begin{aligned} f_x &= \frac{\partial f}{\partial x} = \lim_{h \to \infty} \frac{f(x+h,y) - f(x,y)}{h} \quad \text{and} \\ f_y &= \frac{\partial f}{\partial y} = \lim_{h \to \infty} \frac{f(x,y+h) - f(x,y)}{h}. \end{aligned}$$

In other words, to find the partial derivative of $f$ with respect to $x$ (i.e. $f_x$), simply take the derivative of $f$ treating $x$ as the variable and $y$ as a constant. Similarly, to get the partial derivative with respect to $y$ (i.e. $f_y$), simply take the derivative of $f$ treating $y$ as the variable and $x$ as a constant.

**Observation 94.** The partial derivative with respect to $x$ represents the slope of the tangent lines to the curve that are parallel to the $xz$-plane (i.e. in the direction of $\langle 1, 0, \ldots \rangle$). Similarly, the partial derivative with respect to $y$ represents the slope of the tangent lines to the curve that are parallel to the $yz$-plane (i.e. in the direction of $\langle 0, 1, \ldots \rangle$).

**Problem 95.** If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

**Problem 96.** Let $f(x,y) = \sin\left(\frac{x}{1+y}\right)$. Find the first order partial derivatives of $f(x,y)$.

**Problem 97.** Let $z = f(x, y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ of $x^3 + y^3 + z^3 + 6xyz = 1$.

### 14.3.2 Higher Order Partial Derivatives

**Problem 98.** Find the second order partial derivatives of $f(x, y) = x^3 + x^2y^3 - 2y^2$.

**Theorem 99** (Clairaut)**.** Suppose $f(x, y)$ is defined on a disk $D \subset \mathbb{R}^2$. Then

$$f_{xy} = f_{yx}.$$

**Problem 100.** Calculate $f_{xxyz}(x, y)$ given $f(x, y, z) = \sin(3x + yz)$

Suggested Problems: Section 14.3 : $15 - 44, 47 - 52, 53 - 58, 63 - 66$.

## 14.4 Tangent Planes & Linear Approximation

**Goal 101.** As one zooms into a surface, the more the surface resembles a plane. More Specifically, the surface looks more and more like the tangent plane. Some functions are difficult to evaluate at a point; so, the equation of the tangent plane (which is much simpler) is used to approximate the value of that curve at a given point.

**Theorem 102** (Equation of Tangent Plane)**.** Suppose that $f(x, y)$ has continuous partial derivatives. An equation of the tangent plane (equivalently, the linear approximation) to the surface $z = f(x, y)$ at the point $P = (x_0, y_0, z_0)$ is

$$z - z_0 = f_x\left(x_0, y_0\right)\left(x - x_0\right) + f_y\left(x_0, y_0\right)\left(y - y_0\right).$$

**Problem 103.** Find an equation for the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

**Problem 104.** Give the linear approximation of $f(x, y) = xe^{xy}$ at $(1, 0)$. Then use this to approximate $f(1.1, -0.1)$.

**Problem 105.** Find the linear approximation of $f(x,y) = \sqrt{25 - x^2 - y^2}$ centered at $(-3,0)$ to evaluate $f(-3.04, 0.09)$.

**Problem 106.** Find the linear approximation of the function $f\left(x,y,z\right) = x^3\sqrt{y^2+z^2}$ at the point $(2,3,4)$ and use it to estimate the number $\left(1.98\right)^3\sqrt{\left(3.10\right)^2+\left(3.97\right)^2}$.

Suggested problems: Section $14.4 : 1-6, 11-16$.

# 14.5 Chain Rule

## 14.5.1 Chain Rule

**Theorem 107** (Chain Rule)**.** Suppose that $u$ is a differentiable function of $n$ variables, $x_1, x_3, \ldots, x_n$, each of which has $m$ variables, $t_1, t_2, \ldots, t_m$. Then for each $i \in \{1, 2, \ldots, m\}$,

$$\frac{\partial u}{\partial t_i} = \left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial x_1}{\partial t_i}\right) + \left(\frac{\partial u}{\partial x_2}\right)\left(\frac{\partial x_2}{\partial t_i}\right) + \cdots + \left(\frac{\partial u}{\partial x_n}\right)\left(\frac{\partial x_n}{\partial t_i}\right).$$

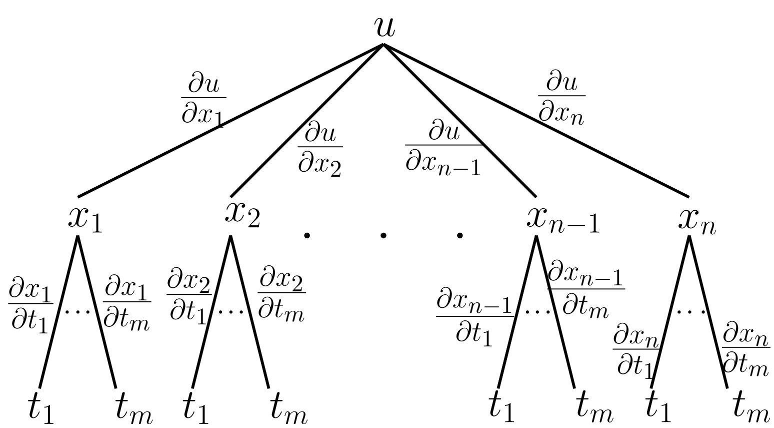


Figure 14.1: Chain Rule

**Problem 108.** Suppose that $z = f(x, y)$ is a differentiable function of $x$ and $y$ where $x = g(t)$ and $y = h(t)$ are both differentiable functions with respect to $t$. Then $z$ is a differentiable function with respect to $t$. Write the formula for $\frac{\partial z}{\partial t}$.

**Problem 109.** If $z = x^2y + 3xy^4$ where $x = \sin(2t)$ and $y = \cos(t)$, find $\frac{\partial z}{\partial t}$ when $t = 0$.

**Problem 110.** Suppose that $z = f(x, y)$ is a differentiable function of $x$ and $y$ where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions with respect to $s$ and $t$. Then $z$ is a differentiable function with respect to $s$ and $t$. Write the formula for $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

**Problem 111.** If $z = e^x \sin(y)$ where $x = st^2$ and $y = s^2 t$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

**Problem 112.** Write the chain rule for $w = f(x, y, z, t)$, where $x = x(u, v), y = y(u, v), z = z(u, v)$, and $t = t(u, v)$. That is, find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

**Problem 113.** If $g(s, t) = f\left(s^2 - t^2, t^2 - s^2\right)$ and $f$ is differentiable, show that $g$ satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

### 14.5.2 Implicit Differentiation

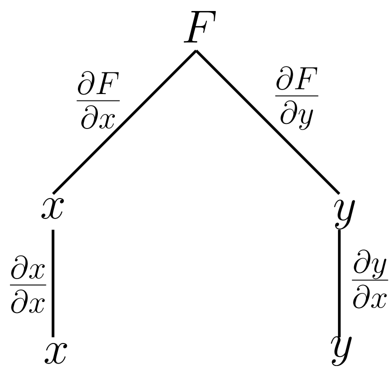


Figure 14.2: Implicit Differentiation

**Problem 114.** Let $F(x, y) = 0$. Show that $\dfrac{\partial y}{\partial x} = -\dfrac{F_x}{F_y}$.

**Problem 115.** Find $y'$ if $x^3 + y^3 = 6xy$.

**Problem 116.** Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz = 1$. *Hint.* $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

Suggested Problems: Section $14.5 : 1 - 12, 17 - 20, 31 - 34$.

## 14.6 Directional Derivatives & Gradient Vector

**Goal 117.** It would be nice to be able to find the slope of the tangent line to a curve $C$ on a surface $S$ in the direction of a unit vector $\vec{\mathbf{u}} = \langle a, b \rangle$.

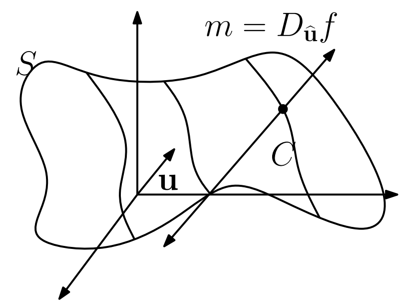


Figure 14.3: Directional Derivative

**Definition 118** (The "Del" Operator)**.** For ease of notation, we denote "del" ($\nabla$) as $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.

**Definition 119** (Gradient)**.** Let $f$ be a differentiable function of two variables, $x$ and $y$. Then the **gradient** of $f$ is the vector function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle .$$

**Theorem 120** (Directional Derivative)**.** If $f$ is a differentiable function of $x$ and $y$, then $f$ has a **directional derivative** in the direction of any unit vector $\widehat{\mathbf{u}} = \langle a, b \rangle$ and

$$D_{\widehat{\mathbf{u}}} f(x, y) = \nabla f(x, y) \cdot \widehat{\mathbf{u}} = f_x(x, y)a + f_y(x, y)b.$$

Equivalently, we can say

$$D_{\widehat{\mathbf{u}}} f(x, y) = f_x(x, y)\cos(\theta) + f_y(x, y)\sin(\theta),$$

where $\theta$ represents the angle of the unit vector.

**Note 121.** The only reason we are restricting the directional derivative to the unit vector is because we care about the rate of change in $f$ per unit distance. Otherwise, the magnitude is irrelevant.

**Problem 122.** If $f(x, y) = \sin(x) + e^{xy}$, find $\nabla f(x, y)$ and $\nabla f(0, 1)$.

**Problem 123.** Find the directional derivative $D_{\vec{\mathbf{u}}}f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ and $\vec{\mathbf{u}}$ is the unit vector given by the angle $\theta = \frac{\pi}{6}$.

**Problem 124.** Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ and in the direction of the vector $\vec{\mathbf{v}} = \langle 2, 5 \rangle$.

**Problem 125.** If $f(x,y,z) = x\sin(yz)$, find the directional derivative of $f(1,3,0)$ in the direction of $\vec{\mathbf{v}} = \langle 1, 2, -1 \rangle$.

Suggested Problems: Section 14.6 : $4 - 17, 21 - 26$.

## 14.7 Maximum & Minimum Values

### 14.7.1 Local Extrema

**Definition 126** (Local Minimum)**.** A function $f$ of two variables $x$ and $y$ has a **local minimum** at the point $(a, b)$ if $f(x, y) \geq f(a, b)$ when $(x, y)$ is near $(a, b)$.

**Definition 127** (Local Maximum)**.** A function $f$ of two variables $x$ and $y$ has a **local minimum** at the point $(a, b)$ if $f(x, y) \leq f(a, b)$ when $(x, y)$ is near $(a, b)$.

**Theorem 128** (First Partial Test)**.** If $f$ has a local extreme at $(a, b)$ and the first order partials exist, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

**Theorem 129** (Second Partial Test)**.** Suppose that the second order partial derivatives of $f$ are continuous on the disc centered at $(a, b)$ and suppose that $f_x(a, b) = f_y(a, b) = 0$. Let $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

- If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- If $D < 0$, then $f(a, b)$ is a saddle point.
- If $D = 0$, then no conclusion can be drawn from this test.

**Note 130.** There is nothing sacred about $f_{xx}$. $D > 0$ means that both $f_{xx}$ and $f_{yy}$ have the same sign. Moreover, we could equivalently check $f_{yy}$ instead of $f_{xx}$ in those cases.

**Problem 131.** Find all local extrema of $f(x, y) = x^4 + y^4 - 4xy + 1$.

**Problem 132.** Find the shortest distance from the point $(1, 0, -2)$ to the plane $x+2y+z = 4$.

**Problem 133.** A topless rectangular box is made from $12\text{m}^2$ of cardboard. Find the dimensions of the box that maximizes the volume of the box.

### 14.7.2 Absolute Extrema

**Theorem 134** (Existence)**.** If $f$ is continuous on a closed and bounded set $D \subset \mathbb{R}^2$, then $f$ attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points $(x_1, y_1)$ and $(x_2, y_2)$ in $D$.

**Strategy 135.** To find absolute extrema,

1. Find critical points and values of $f$ at those critical points
2. Find the extreme values that occur on the boundary. Find the values of $f$ at those points.
3. Compare all of those values for the largest and smallest values.

**Problem 136.** Find the absolute extrema of $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $R = [0, 3] \times [0, 2] = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Suggested Problems: Section $14.7 : 5 - 18, 29 - 36$.

## 14.8 Lagrange Multipliers

**Goal 137.** To have a way of finding absolute extrema of a function that is subject to certain constraints.

**Strategy 138** (Lagrange Multipliers)**.** Let $k \in \mathbb{R}$. To find the maximum and minimum values of $f(x, y, z)$ subject to $g(x, y, z) = k$,

- Find all values of $x, y, z$, and $\lambda$ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$.
- Evaluate all of these points to find the maximum and minimum values.

**Problem 139.** A topless rectangular box is made from $12\text{m}^2$ of cardboard. Find the dimensions of the box that maximizes the volume of the box.

**Problem 140.** Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

**Problem 141.** Find the maximum value of $f(x, y, z) = x + 2y + 3z$ on the curve of the intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Suggested Problems: Section 14.8 : $3 - 18$

# Chapter 15

# Multiple Integrals

## 15.1 Double Integrals over a Rectangle

**Recall 142.** We know the following from Calculus I:

1. We defined the integral in terms of Riemann Sums.
2. That is, we found the area underneath the curve $y = f(x)$ by dividing the area into rectangles. We then added up their areas to get the area under the curve.
3. We then found the exact area of this by evaluating $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$.

### 15.1.1 Volumes as Double Integrals

Let $R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, c \le y \le d\}$ define a rectangle.
Let $S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le f(x,y), (x,y) \in R\}$ define the solid that lies above $R$.

**Goal 143.** We want to divide $R$ into rectangular prisms (show boxes) with the goal of adding up all of their volumes to give us the volume underneath the curve $f(x,y)$.

Let $(x_{ij}^*, y_{ij}^*)$ denote a **sample point** in each division $R_{ij}$.
Let $\Delta A = \Delta x\,\Delta y$ denote the area of each $R_{ij}$. Then we can express this volume as

$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

**Definition 144** (Double Integral)**.** The **Double Integral** of $f$ over the rectangle $R$ is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

We can simplify this if we choose each sample point, $(x_{ij}^*, y_{ij}^*)$, to be the point in the upper right corner of each sub-rectangle. Call this point $(x_i, y_j)$. Then we get:

$$V = \iint\limits_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j)\,\Delta A.$$

**Problem 145.** Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide $R$ into four equal squares and choose the sample point to be the upper right corner of each square $R_{ij}$. Approximate the Volume.

**Problem 146.** Estimate the volume of the solid that lies above the square $R = [0,2] \times [1,2]$ and below the function $x - 3y^2$. Divide $R$ into four equal rectangles and choose the sample point to be the midpoint of each rectangle $R_{ij}$. Approximate the volume.

Suggested Homework: p. 981 exercises 1-5, 11-13.

## 15.2 Iterated Integrals

Okay; so, taking these Riemann Sums is a bit of a pain.

**Recall 147.** In Calculus I, we equated these Riemann sums to the definition of an integral. We will attempt to do the same thing here; however, we will be using two partial integrals to do this.

Suppose that $f$ is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

**Definition 148** (Partial Integral w.r.t. $y$)**.** We define $\int_c^d f(x, y)\, dy$ as the **partial integral** with respect to $y$. We evaluate this integral by treating $x$ as a constant, and integrate $f(x, y)$ with respect to $y$. We can then use the *fundamental theorem of calculus part 2* (theorem 224) to evaluate the integral at on the interval $[c, d]$.

**Definition 149** (Partial Integral w.r.t. $x$)**.** We define $\int_a^b f(x, y)\, dx$ as the **partial integral** with respect to $x$. We evaluate this integral by treating $y$ as a constant, and integrate $f(x, y)$ with respect to $x$. We can then use the *fundamental theorem of calculus part 2* to evaluate the integral at on the interval $[a, b]$.

**Definition 150** (Double Integral)**.** The double integral is defined as follows:

$$\int_a^b \int_c^d f(x, y)\, dx\, dy = \int_a^b \left[ \int_c^d f(x, y)\, dx \right] dy.$$

In other words, we work this integral from the *inside out*.

**Problem 151.** Evaluate the integrals (a) $\displaystyle\int_0^3 \int_1^2 x^2 y\, dy\, dx$, and (b) $\displaystyle\int_1^2 \int_0^3 x^2 y\, dx\, dy$.

**Theorem 152** (Fubini)**.** If $f$ is continuous on the rectangle $R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

**Problem 153.** Evaluate the double integral $\iint\limits_R (x - 3y^2)\,dA$, where $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

**Problem 154.** Evaluate $\iint\limits_R y \sin(xy)\,dA$, where $R = [1, 2] \times [0, \pi]$.

**Problem 155.** Find the volume of the solid $S$ that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

**Problem 156.** Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

Suggested Homework p. 987, 1-20, 25-30

## 15.3 Double Integrals over General Regions

**Question 157.** Okay, So now we know how to find the volume of the region under a surface given that the the projection of the region down to the $xy$-plane is rectangular. What if that region is defined as the boundary between two functions?

### 15.3.1 Type 1

Type I regions are regions of the form $R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$. That is to say that the region in the $xy$-plane looks like this:

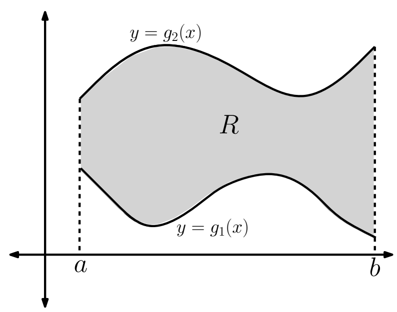


Figure 15.1: Type 1 Double Integral

Moreover, if $f$ if continuous on a Type I region, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

### 15.3.2 Type 2

Type II regions are regions of the form $R = \{(x, y) \in \mathbb{R}^2 \mid h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$. That is to say that the region in the $xy$-plane looks like this:

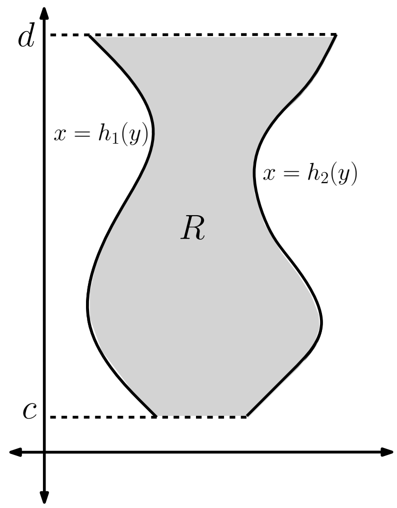


Figure 15.2: Type 2 Double Integral

Moreover, if $f$ if continuous on a Type I region, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

**Problem 158.** Evaluate $\displaystyle\iint_D (x+2y)\,dA$, where $D$ is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

**Problem 159.** Evaluate $\displaystyle\iint_D xy\,dA$, where $D$ is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

**Problem 160.** Show that $\displaystyle\iint_D 1\,dA = A_D$, where $A_D$ denotes the area of the region $D$.

Suggested Homework: p 995, 1-10, 17-32

## 15.10 Change of Variables in Multiple Integrals

We have done changes of variables several times in the past. Dating as far back as Calculus I when we learned $u$ -substitution, we started using changes of variables (we made $u = g(x)$.)

**Goal 161.** The goal of this section is to write a general form for a change of variables. In other words, is there a transformation on the function we can do to make the integral easier.

**Definition 162** (Transformation)**.** A change of variables is a **transformation** , $T$, from the $uv$-plane to the $xy$-plane, $T(u, v) = (x, y)$, where $x$ and $y$ are related to $u$ and $v$ by the equations

$$x = g(u, v) \qquad y = h(u, v).$$

We usually take these transformations to be $C^1$-**Transformation**, meaning $g$ and $h$ have continuous first-order partial derivatives.

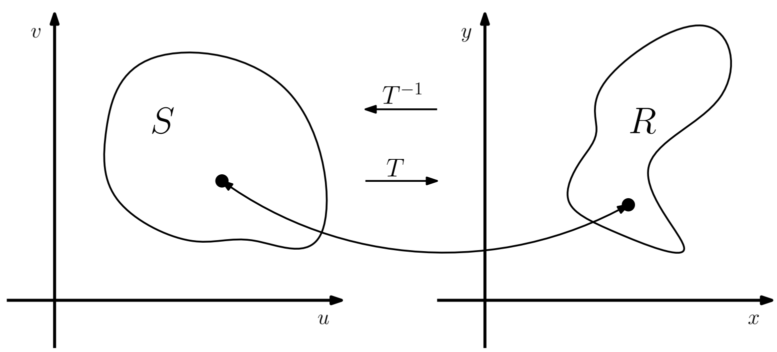


Figure 15.3: Transformation

**Definition 163** (Jacobian)**.** The **Jacobian** of the transformation $T$ given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) - \left(\frac{\partial x}{\partial v}\right)\left(\frac{\partial y}{\partial u}\right).$$

**Theorem 164** (Double Integral Change of Variable)**.** Suppose that $T$ is a $C^1$-transformation whose Jacobian is nonzero, and suppose that $T$ maps a region $S$ int he $uv$-plane onto a region $R$ in the $xy$-plane. Let $f$ be a continuous function on $R$. Suppose also that $T$ is a one-to-one transformation except perhaps along the boundary of the regions. Then

$$\iint\limits_R f(x, y)\, dA = \iint\limits_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv.$$

**Problem 165.** Evaluate $\iint\limits_R x + y\,dA$ where $R$ is the trapezoidal region with vertices given by $(0,0), (5,0), \left(\frac{5}{2}, \frac{5}{2}\right)$, and $\left(\frac{5}{2}, -\frac{5}{2}\right)$ using the transformation $x = 2u + 3v$ and $y = 2u - 3v$.

**Problem 166.** Evaluate the integral $\iint\limits_R e^{(x+y)/(x-y)}\,dA$, where $R$ is the trapezoidal region with vertices $(1,0), (2,0), (0,-2)$, and $(0,-1)$.

Suggested Homework: p. 1047, 15-20

# 15.4 Double Integrals in Polar Coordinates

## 15.4.1 Crash Course in Polar Coordinates

We have spent most of our lives in the Cartesian Coordinate System, which was invented by none other than René Descartes, who was because he thought. Sometimes, however, functions (and consequently integrals) become simpler when expressed in different coordinate systems. There are many different coordinate systems. Today, however, we will focus on one that was invented by Sir Isaac Newton – the Polar Coordinate System.

**Definition 167** (Origin & polar axis)**.** First, we must pick a special point in the plane – the **origin**. Once we have the origin, draw a ray from the origin in any direction. This ray is called the **polar axis**.

**Definition 168** (Points)**.** Points in this system are defined by two parameters $(r, \theta)$, where $r$ is the distance the point is from the origin and $\theta$ is the angle between the polar axis and the line that connects the point to the origin. Since a picture is worth a thousand words, here is a picture describing what was just described:

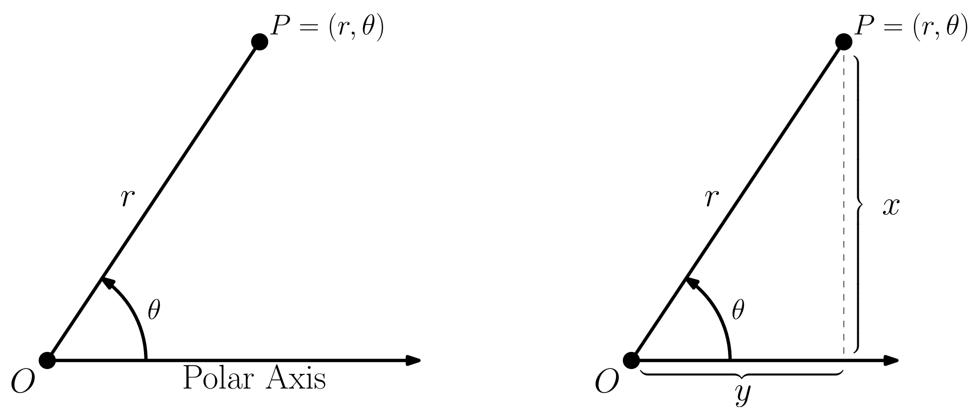


Figure 15.4: Point in Polar Coordinates

Naturally, there is a translation from Polar Coordinates to the Cartesian Coordinate system and vice versa. That translation looks like:

$$x = r\cos(\theta), \qquad y = r\sin(\theta). \qquad\qquad \text{Moreover,}$$
$$r^2 = x^2 + y^2, \qquad \tan(\theta) = \tfrac{y}{x}$$

**Definition 169** (Graphs in Polar Coordinates)**.** The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points $P$ that have at least one polar representation $(r, \theta)$ whose coordinates satisfy the equation.

**Example 170.** The graph representing the polar equation $r = 2$ is

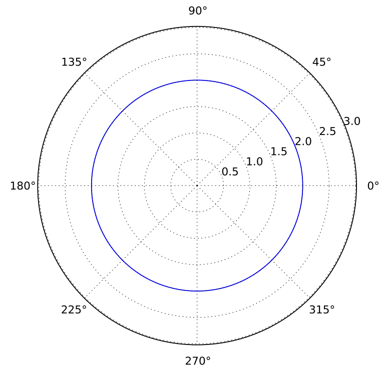


Figure 15.5: $r = 2$

**Example 171.** The graph representing the polar equation $\theta = 1$ is

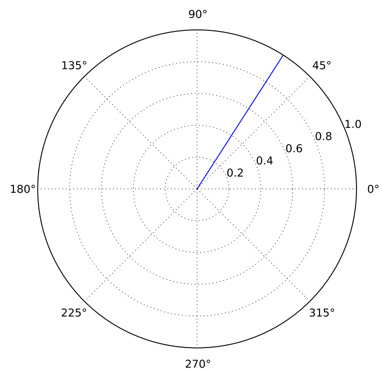


Figure 15.6: $\theta = 1$

**Example 172.** The graph representing the polar equation $r = 2\cos(\theta)$ is

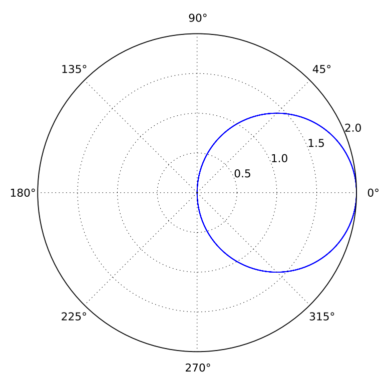


Figure 15.7: $r = 2\cos(\theta)$

**Example 173.** The graph representing the polar equation $r = 1 + \sin(\theta)$ is

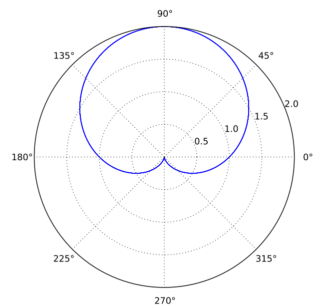


Figure 15.8: $r = 1 + \sin(\theta)$

**Example 174.** The graph representing the polar equation $r = \cos(2\theta)$ is

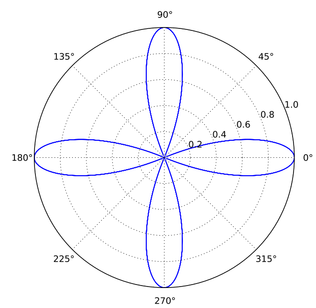


Figure 15.9: $r = \cos(2\theta)$

## 15.4.2 Double Integrals with Polar Coordinates

In the polar coordinate system, a rectangle in the $r\theta$-plane can be defined as

$$R = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}.$$

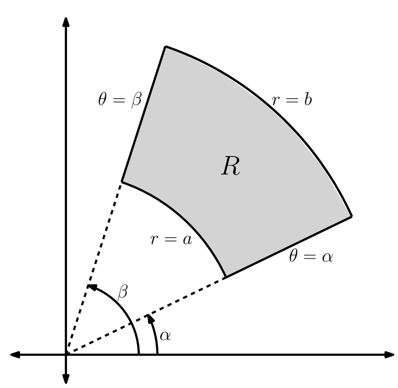


Figure 15.10: Rectangle in Polar Coordinates

We can divide up this rectangle just as we did in section 15.1, dividing the rectangle into tiny rectangular prisms and summing up their volumes. Moreover, we can generalize Fubini's theorem (theorem 152) as follows:

**Problem 175.** Show that when dealing with polar coordinates, $dA = r\,dr\,d\theta$.

**Theorem 176** (Polar version of Fubini's Theorem)**.** If $f$ is continuous on the polar rectangle $R$ given by $0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint\limits_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$$

**Note:** Do not forget the $r$ before the $dr\,d\theta$!

**Problem 177.** Evaluate $\displaystyle\iint\limits_R (3x+4y^2)\,dA$, where $R$ is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

**Problem 178.** Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

**Theorem 179** (Polar version of section 15.3.2)**.** If $f$ is continuous on a polar region of the form $D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$, then

$$\iint\limits_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

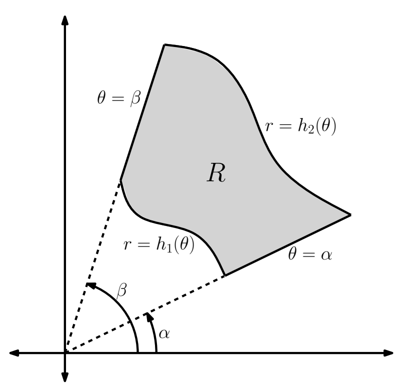


Figure 15.11: Polar version of section 15.3.2

**Problem 180.** Use a double integral to find the area enclosed by one loop of the four leaved rose $r = \cos(2\theta)$.

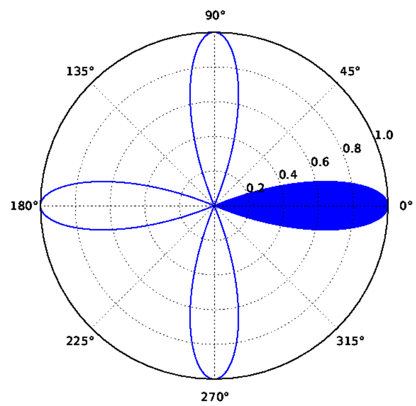


Figure 15.12: One Rose of $r = \cos(2\theta)$

**Problem 181.** Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the $xy$-plane, and inside the cylinder $x^2 + y^2 = 2x$.

*Hint.*

- First, find what the "rectangle" in polar coordinates looks like.
- That is to say, translate $x^2 + y^2 = 2x$ into polar coordinates and see what that region looks like.
- This will be your "rectangle."
- Then, look at $z = x^2 + y^2$ as a polar function.
- Use this as your integrand.
- Evaluate.
- Don't forget the $r$ in "$r\,dr\,d\theta$"!

Suggested Homework: p. 1002, 7-27 odd.

## 15.7 Triple Integrals

**Remark 182.** The interpretation of the triple integral $\iiint\limits_E f(x,y,z)\,dV$, where $f(x,y,z) \geq 0$ is the "hyper-volume" of a four-dimensional object, which is interesting, but admittedly not very useful. Of course, $E$ represents a subset of the domain of $f$, which lives in fourth-dimensional space.

**Remark 183.** The triple integral $\iiint\limits_E f(x,y,z)\,dV$ can be interpreted in different ways in different (useful) physical situations, depending on the interpretation of $x, y$, and $z$ as well as what $f(x,y,z)$ represents.

**Theorem 184** (Generalization of Problem 160)**.** The volume $V(E)$ of a subset $E$ of $\mathbb{R}^3$ is can be evaluated by the following:

$$V(E) = \iiint\limits_E 1\,dV.$$

**Problem 185.** Use the triple integral to find the volume of the tetrahedron $T$ bounded by the planes $x + 2y + z = 2, x = 2y, x = 0$, and $z = 0$.

Suggested Homework: p. 1025, 3-8, 9-11.

## 15.8 Triple Integrals in Cylindrical Coordinates

**Definition 186** (Points in Cylindrical Coordinates)**.** In the cylindrical coordinate system, a **point** $P$ in three-dimensional space is represented as an ordered triple $(r, \theta, z)$, where $r$ and $\theta$ are polar coordinates of the projection of $P$ onto the $xy$-plane and $z$ is the directed distance from the $xy$-plane to $P$.

**Theorem 187** (Conversion: Cylindrical $\leftrightarrow$ Rectangular)**.**

$$\begin{array}{llll} \text{Cylindrical} \to \text{Rectangular:} & x = r\cos(\theta) & y = r\sin(\theta) & z = z \\ \text{Rectangular} \to \text{Cylindrical:} & r^2 = x^2 + y^2 & \tan(\theta) = \left(\frac{y}{x}\right) & z = z \end{array}$$

**Theorem 188** (Cylindrical Version of Fubini's Theorem)**.** Let $E = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$ be a region in $\mathbb{R}^3$ whose projection down to the $xy$-plane, $D$, is a region that satisfies theorem 176. That is, $D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$. Then

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta), r\sin(\theta))}^{u_2(r\cos(\theta), r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)\, r\, dz\, dr\, d\theta.$$

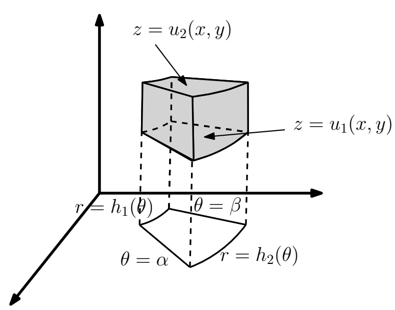


Figure 15.13: Cylindrical Version of Fubini's Theorem

**Problem 189.** (a) Plot the point with cylindrical coordinates $(2, \frac{2\pi}{3}, 1)$ and find its rectangular coordinates.
(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, 7)$.

**Problem 190.** Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$.
*Hint.* Change the triple integral into cylindrical coordinates.

Suggested Homework: p. 1031, 1-4, 7-24.

## 15.9 Triple Integrals in Spherical Coordinates

**Definition 191** (Spherical Coordinates)**.** A point $P$ in $\mathbb{R}^3$ in **spherical coordinates** is denoted by $(\rho, \theta, \phi)$, where $\rho$ is the length of the line segment from the origin, $O$, to $P$ (denoted $\overline{OP}$), $\theta$ is the angle from the $x$-axis to the line segment $\overline{OP}$ projected down to the $x$-axis, and $\phi$ is the angle between the positive $z$-axis and the line segment $\overline{OP}$.

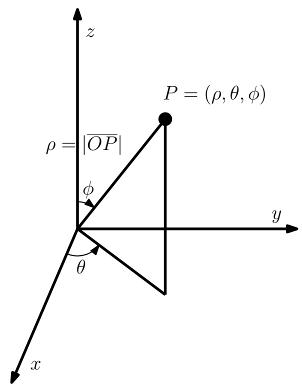


Figure 15.14: Point in Spherical Coordinates

**Note 192.** By observing this definition, we can see the following inequalities:

$$\rho \geq 0 \qquad 0 \leq \phi \leq \pi \qquad 0 \leq \theta \leq 2\pi.$$

**Definition 193** (Spherical to Rectangular)**.** To convert from Spherical to Rectangular coordinates, the following equations can be used:

$$x = \rho \sin(\phi)\cos(\theta) \qquad y = \rho \sin(\phi)\sin(\theta) \qquad z = \rho\cos(\phi).$$

Moreover,

$$\rho^2 = x^2 + y^2 + z^2 \qquad \theta = \arctan\left(\frac{y}{x}\right) \qquad \phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right).$$

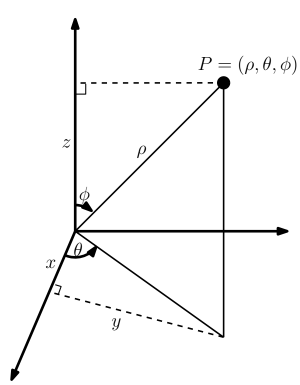


Figure 15.15: Spherical Coordinates to Rectangular Coordinates

**Definition 194** (Higher order Jacobian)**.** The **Jacobian** of $T$ is the following determinant (recall determinants from doing the cross product):

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial 2} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial 2} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial 2} \end{vmatrix}.$$

**Theorem 195** (Triple Integral Change of Variables)**.** Suppose that $T$ is a $C^1$-transformation whose Jacobian is nonzero, and suppose that $T$ maps a region $S$ int he $uv$-plane onto a region $R$ in the $xy$-plane. Let $f$ be a continuous function on $R$. Suppose also that $T$ is a one-to-one transformation except perhaps along the boundary of the regions. Then

$$\iiint\limits_R f(x,y,z)\,dV = \iiint\limits_S f\left(x\left(u,v,w\right), y\left(u,v,w\right), z\left(u,v,w\right)\right) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

**Problem 196.** Show that when dealing with spherical coordinates, $dV = \rho^2 \sin(\phi)\,d\rho\,d\theta\,d\phi$. *Recall.* $x = \rho\sin(\phi)\cos(\theta), y = \rho\sin(\phi)\sin(\theta)$, and $z = \rho\cos(\phi)$.

**Theorem 197** (Triple Integral with Spherical Coordinates)**.**

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f\left(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)\right)\, \rho^2\, \sin(\phi)\,d\rho\,d\theta\,d\phi,$$

where $E$ is a spherical wedge given by

$$E = \{(\rho,\theta,\phi) \in \mathbb{R}^3 \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}.$$

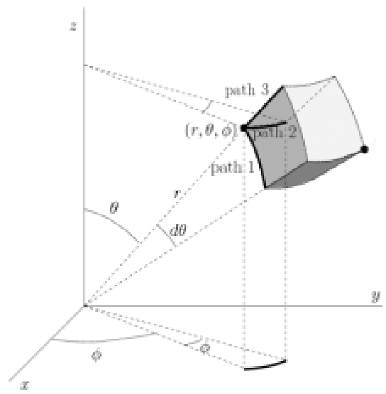


Figure 15.16: Spherical Triple Integral

**Note 198.** Observe that $dV = \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$.

**Theorem 199** (Spherical Fubini's Theorem)**.** We can extend the theorem 197 to regions defined by

$$E = \{(\rho,\theta,\phi) \in \mathbb{R}^3 \mid g_1(\theta,\phi) \leq \rho \leq g_2(\theta,\phi), \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

in such a way:

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f\left(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)\right)\, \rho^2\, \sin(\phi)\,d\rho\,d\theta\,d\phi,$$

**Problem 200.** Evaluate $\iiint\limits_B e^{\left(x^2+y^2+z^2\right)^{3/2}}\,dV$, where $B$ is the unit ball

$$B = \left\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\right\}.$$

**Problem 201.** Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$

Suggested Homework: p. 1037, 17,18,21-26

# Chapter 16

# Vector Calculus

## 16.1 Vector Fields

**Definition 202** (Two-Dimensional Vector Field)**.** Let $D$ be a set in $\mathbb{R}^2$. A **vector field** of $\mathbb{R}^2$ is a function $\overrightarrow{\mathbf{F}}$ that assigns to each point $(x, y)$ in $D$ a two-dimensional vector $\overrightarrow{\mathbf{F}}(x, y)$.

**Definition 203** ($n$-Dimensional Vector Field)**.** Let $E$ be a set in $\mathbb{R}^n$. A **vector field** of $\mathbb{R}^n$ is a function $\overrightarrow{\mathbf{F}}$ that assigns to each point $(x_1, x_2, \ldots, x_n)$ in $E$ an $n$-dimensional vector $\overrightarrow{\mathbf{F}}(x_1, x_2, \ldots, x_n)$.

**Problem 204.** A vector field on $\mathbb{R}^2$ is defined by $\overrightarrow{\mathbf{F}}(x, y) = \langle -y, x\rangle$. Describe $\overrightarrow{\mathbf{F}}$ by sketching some of the vectors $\overrightarrow{\mathbf{F}}(x, y)$.

**Problem 205.** Show that each vector defined in the vector field from Problem 204 is tangent to a circle with center at the origin. *Hint.* Let $\vec{\mathbf{x}} = \langle x, y \rangle$ (this is called the position vector). Use the dot product to show that they are perpendicular.

**Problem 206.** Sketch the vector field of $\mathbb{R}^3$ given by $\vec{\mathbf{F}}(x, y, z) = \langle 0, 0, z \rangle$.

**Definition 207** (Gradient Field)**.** Since the gradient of a function is the vector of partial derivatives, it is really a vector field called the **gradient vector field**. Namely, if $f(x, y)$ is a function of two variables, $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$ is a gradient vector field of $\mathbb{R}^2$. Similarly, if $f$ is a function of three variables, $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$ is a gradient vector field of $\mathbb{R}^3$.

**Problem 208.** Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of $f$. How are they related?

Suggested Problems: p. 1061, 1-14

## 16.2 Line Integrals

Up to this point, our intervals of integration were always either bijective function or a closed interval $[a, b]$. In this section, we will be integrating over a parametrized curve instead of a nice interval as before.

### 16.2.1 Line Integrals in the Plane

**Goal 209.** To integrate functions along a curve as opposed to along an interval.

**Recall 210** (Arc Length)**.** The length along a curve $C$ is $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

**Definition 211** (Line Integral with respect to Arc Length)**.** If $f$ is defined on a smooth curve $C$ (parametric equation with respect to $t$), then the **line integral of $f$ along $C$ in $\mathbb{R}^2$** is

$$\int\limits_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

**Problem 212.** Evaluate $\int\limits_C (2+x^2y)\,ds$, where $C$ is the upper half of the unit circle $x^2+y^2 = 1$.

**True Fact 213.** If $C$ is the union of finitely many smooth surves $C_1, C_2, \ldots, C_n$, then

$$\int\limits_C f(x,y)\,ds = \int\limits_{C_1} f(x,y)\,ds + \int\limits_{C_2} f(x,y)\,ds + \cdots + \int\limits_{C_n} f(x,y)\,ds.$$

**Problem 214.** Evaluate $\int\limits_C 2x\,ds$, where $C$ consists of the arc $C_1$ of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment $C_2$ from $(1,1)$ to $(1,2)$.

**Definition 215** (Line Integral with respect to $x$ and $y$)**.** Let $x = x(t), y = y(t), dx = x'(t)\,dt$, and $dy = y'(t)\,dt$. Then the **line integral with respect to $x$ and $y$** are respectively:

$$\begin{aligned}\int\limits_C f(x,y)\,dx &= \int_a^b f(x(t),y(t))x'\,dt\\ \int\limits_C f(x,y)\,dy &= \int_a^b f(x(t),y(t))y'\,dt\end{aligned}$$

**Problem 216.** Evaluate $\int\limits_C y^2\,dx + x\,dy$, where

- $C$ is the line segment from $(-5,-3)$ to $(0,2)$. *Recall.* The vector representation of the line segment starting at $\vec{\mathbf{r}}_0$ and ending at $\overrightarrow{\mathbf{r_1}}$ is given by $\vec{\mathbf{r}}(t) = (1-t)\vec{\mathbf{r}}_0 + t\vec{\mathbf{r}}_1$.

- $C$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

### 16.2.2 Line Integrals in Space

First, the definition for the line integral with respect to arc length (Definition 211) can be generalized as follows:

**Definition 217.** (Line Integral with respect to Arc Length) If $f$ is defined on a smooth curve $C$ (parametric equation with respect to $t$), then the **line integral of $f$ along $C$ in $\mathbb{R}^3$** is

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt.$$

**Problem 218.** Evaluate $\int_C y\sin(z)\,ds$, where $C$ is the circular helix given by the equation $x=\cos(t), y=\sin(t), z=t, 0\le t\le 2\pi$.

**Problem 219.** Evaluate $\int_C y\,dx+z\,dy+x\,dz$, where $C$ consists of the line segment $C_1$ from $(2,0,0)$ to $(3,4,5)$, followed by the vertical line segment $C_2$ from $(3,4,5)$ to $(3,4,0)$.

### 16.2.3 Line Integrals of Vector Fields

**Definition 220** (Line Integral of Vector Field)**.** Let $\vec{\mathbf{F}}$ be a continuous vector field defined on a smooth curve $C$ given by a vector function $\vec{\mathbf{r}}(t), a \leq t \leq b$. Then the **line integral of $\vec{\mathbf{F}}$ along** $C$ is

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = W.$$

We say that work is the line integral with respect to arc length of the tangential component of force.

**Problem 221.** Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y, z) = \langle xy, yz, zx \rangle$ and $C$ is the twisted cubic given by $x = t, y = t^2, z = t^3, 0 \leq t \leq 1$.

**Theorem 222** (Equivalent to definition 220)**.** Let $\vec{\mathbf{F}} = \langle P, Q, R \rangle$. Then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P\, dx + Q\, dy + R\, dz.$$

**Problem 223.** Take another look at problem 219. Express problem 219 as $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y, z) = \langle y, z, x \rangle$ .

Suggested Homework: p. 1072, 1-16, 19-22

## 16.3 The Fundamental Theorem of Line Integrals

**Recall 224** (Fundamental Theorem of Calculus part II)**.** If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F$ is *any* antiderivative of $f$, that is, a function such that $F' = f$.

**Goal 225.** It would be nice to get a generalization of the Fundamental Theorem of Calculus Part II (224) to line integrals.

**Theorem 226** (Fundamental Theorem of Line Integrals)**.** Let $C$ be a smooth curve given by the vector function $\vec{\mathbf{r}}(t), a \leq t \leq b$. Let $f$ be a differentiable function of two or three variables whose gradient vector $\nabla f$ is continuous on $C$. Then

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).$$

**Definition 227** (Conservative Vector Field & Potential Functions)**.** $\vec{\mathbf{F}}$ is a **conservative vector field** if there is a function $f$ such that $\vec{\mathbf{F}} = \nabla f$. The function $f$ is called a **potential function** for the vector field.

**Observation 228** (Independence of Path)**.** In general, $\int\limits_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int\limits_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$; however, theorem 226 says that $\int\limits_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int\limits_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ whenever $\vec{\mathbf{F}}$ is a conservative vector field! Thus, we can say that line integrals of conservative vector fields are independent of path.

**Problem 229.** Find a function $f$ such that $\vec{\mathbf{F}}(x, y) = \langle x^2, y^2 \rangle = \nabla f$ and use this to evaluate $\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ along the arc $C$ of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$.

**Definition 230** (Closed Curve)**.** A curve is called **closed** if its terminal point coincides with its initial point. That is to say that $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$.

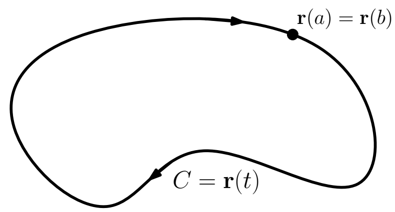


Figure 16.1: Closed Curve

**Theorem 231.** $\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is independent of path in $D$ if and only if $\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$ for every closed path $C$ in $D$.

**Definition 232** (Open Set)**.** A set $D$ is said to be **open** if every point $P$ in $D$ has a disk with center $P$ that is contained wholly and solely in $D$. *Note.* $D$ cannot contain any boundary points.

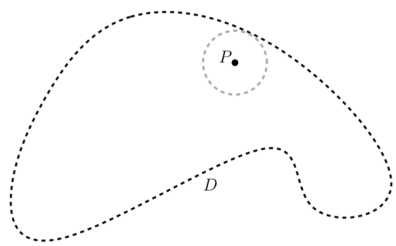


Figure 16.2: Open Set

**Definition 233** (Connected Set)**.** A set $D$ is said to be **connected** if for every two points $P$ and $Q$ in $D$, there exists a path which connects $P$ to $Q$.

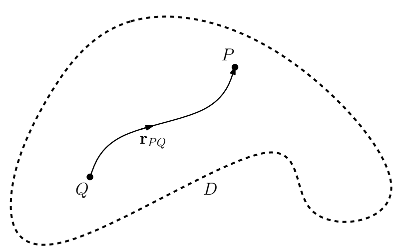


Figure 16.3: Connected Set

**Theorem 234.** Suppose that $\vec{\mathbf{F}}$ is a vector field that is continuous on an open connected region $D$. If $\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is independent of path in $D$, then $\vec{\mathbf{F}}$ is a conservative vector field on $D$. That is to say that there exists a function $f$ such that $\nabla f = \vec{\mathbf{F}}$.

**Theorem 235** (Clairaut's Theorem for Conservative Vector Fields)**.** If $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a conservative vector field, where $P$ and $Q$ have continuous first order partial derivatives on a domain $D$, then throughout $D$, we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

**Definition 236** (Simply-Connected)**.** A **simply-connected** region in the plane is a connected region $D$ such that every simple closed curve in $D$ encloses only points that are in $D$. $D_1$ in figure 16.3 is the only simply connected region. *Note.* $D_1$ has no holes and is not two disconnected regions.

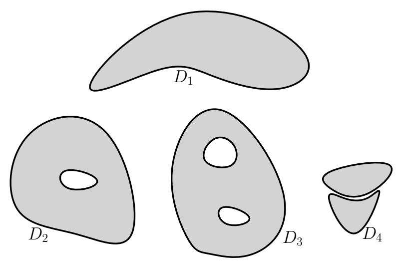


Figure 16.4: Simply-Connected Set

**Theorem 237** (Consequence of Green's Theorem)**.** Let $\vec{\mathbf{F}} = \langle P, Q\rangle$ be a vector field on an open simply-connected region $D$. Suppose that $P$ and $Q$ have continuous first order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{Throughout } D.$$

Then, $\vec{\mathbf{F}}$ is a conservative vector field. There will be more on this to come. It is a special case of Green's Theorem which we will see in section 16.4

**Problem 238.** Determine whether or not the vector field $\vec{\mathbf{F}}(x,y) = \langle x-y, x-2\rangle$ is a conservative vector field.

**Problem 239.** Determine whether or not the vector field $\vec{\mathbf{F}}(x, y) = \langle 3 + 2xy, x^2 - 3y^2 \rangle$ is a conservative vector field.

**Problem 240.** If $\vec{\mathbf{F}}(x, y, z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$ , find a function $f$ such that $\nabla f = \vec{\mathbf{F}}$.

Suggested Problems: p. 1082, 3-10, 12-18

## 16.4 Green's Theorem

**Definition 241** (Orientation)**.** Traversing a curve $C$ in a counterclockwise direction is said to be a **positive orientation** of the curve $C$. Similarly, traversing a curve $C$ in a clockwise direction is said to be a **negative orientation** of a curve $C$.

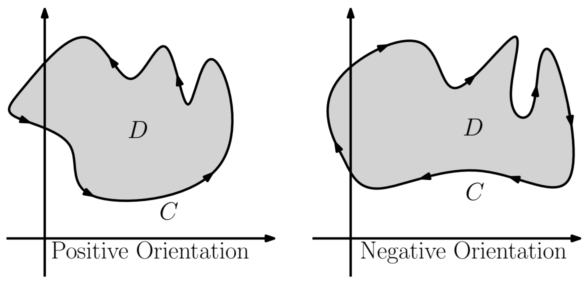


Figure 16.5: Orientation

**Theorem 242** (Green's Theorem)**.** Let $C$ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

**Notation 243.** We denote the line integral calculated by using the positive orientation of the closed curve $C$ by $\oint_C P\,dx + Q\,dy$, $\oint_C P\,dx + Q\,dy$, or $\oint_C P\,dx + Q\,dy$,. We denote line integrals calculated by using the negative orientation of the closed curve $C$ by $\oint_C P\,dx + Q\,dy$,

**Theorem 244** (Area Using Green's theorem)**.** The area of a region $D$ can be calculated using Green's Theorem as follows:

$$A_D = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

**Problem 245.** Evaluate $\int\limits_C x^4\,dx + xy\,dy$, where $C$ is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

**Problem 246.** Evaluate $\oint\limits_C \left(3y - e^{\sin(x)}\right)\,dx + \left(7x + \sqrt{y^4+1}\right)\,dy$, where $C$ is the circle $x^2 + y^2 = 9$.

**Problem 247.** Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. *Hint.* The ellipse has a parametric equation $x = a\cos(t), y = b\sin(t)$, where $0 \le t \le 2\pi$.

**Theorem 248** (Extended Green's Theorem)**.** Let $D = D_1 \cup D_2$, where $D_1$ and $D_2$ are both simple. Let $C = C_1 \cup C_2$ denote the boundary of $D$. Then,

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

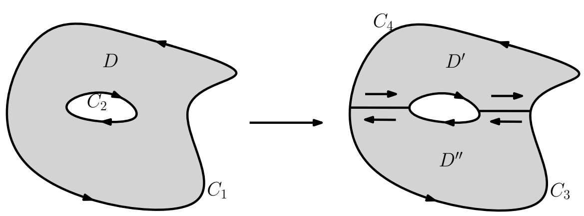


Figure 16.6: Generalization of Green's Theorem

**Problem 249.** Evaluate $\oint y^2\,dx + 3xy\,dy$, where $C$ is the boundary of the semiannual region $D$ in the upper half-plane between the circle $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. *Hint.* Use Polar Coordinates.

Suggested Problems: p. 1089, 5-14.

# 16.5 Curl and Divergence

## 16.5.1 Curl

We now define a the curl of a function, which helps us represent rotations of different sorts in physics and such fields. It can be used, for instance, to represent the velocity field in fluid flow.

**Recall 250** (Del)**.** For ease of notation, we denote "del" ($\nabla$) as $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.

**Definition 251** (Curl)**.** Let $\overrightharpoon{\mathbf{F}} = \langle P, Q, R \rangle$ be a vector field in $\mathbb{R}^3$. The curl of $\overrightharpoon{\mathbf{F}}$ is defined as:

$$\begin{aligned}
\text{curl } \overrightharpoon{\mathbf{F}} &= \nabla \times \overrightharpoon{\mathbf{F}} \\
&= \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \left\langle \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right), \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right), \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right\rangle.
\end{aligned}$$

**Problem 252.** If $\overrightharpoon{\mathbf{F}}(x, y, z) = \langle xz, xyz, -y^2 \rangle$, find the curl $\overrightharpoon{\mathbf{F}}$.

**Theorem 253.** If $f$ is a function of three variables that has continuous second order partial derivatives, then curl $(\nabla f) = \vec{\mathbf{0}}$.

**Problem 254.** Prove theorem 253.

**Theorem 255.** If $\vec{\mathbf{F}}$ is a vector field defined on all $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is a conservative vector field.

**Problem 256.** Show that the vector field $\vec{\mathbf{F}}(x, y, z) = \langle xz, xyz, -y^2 \rangle$ is not conservative.

### 16.5.2 Divergence

The divergence can be understood once again in terms of fluid flow. If $\vec{\mathbf{F}}$ is the velocity of a fluid, then the divergence of $\vec{\mathbf{F}}$ represents the net rate of change with respect to time of the mass of the fluid per unit volume.

**Definition 257** (Divergence)**.** If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$, then the **divergence**, div $\vec{\mathbf{F}}$, is defined as

$$\text{div } \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

**Problem 258.** If $\vec{\mathbf{F}}(x, y, z) = \langle xz, xyz, -y^2\rangle$ , find div $\vec{\mathbf{F}}$.

**Theorem 259.** If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is a vector field in $\mathbb{R}^3$ and $P, Q$, and $R$ have continuous second order partial derivatives, then

$$\text{div}\left(\text{curl } \vec{\mathbf{F}}\right) = 0.$$

**Problem 260.** Prove theorem 259

### 16.5.3 Vector Forms of Green's Theorem

Green's theorem as we have studied in section 16.4 can be viewed as the line integral of the tangential component of $\vec{\mathbf{F}}$ along the curve $C$ as the double integral of the vertical components of the curl $\vec{\mathbf{F}}$ over the region $D$ enclosed by $C$. That is to say,

**Theorem 261** (Line Integral of Tangential Component)**.** Let $C$ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then the line integral of the tangential component of $\vec{\mathbf{F}}$ along a curve $C$ is

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D \left(\text{curl } \vec{\mathbf{F}}\right) \cdot \vec{\mathbf{k}}\, dA.$$

A similar thing can be said for the line integral of the normal component of $\vec{\mathbf{F}}$ along the curve $C$.

**Definition 262** (Outward Unit Normal Vector)**.** Let $\vec{\mathbf{r}}(t)$ denote the vector equation of the curve $C$. That is, $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle, a \leq t \leq b$.The **outward unit normal vector**, $\vec{\mathbf{n}}(t)$ is defined as

$$\vec{\mathbf{n}}(t) = \left\langle \frac{y'(t)}{|\vec{\mathbf{r}}'(t)|}, -\frac{x'(t)}{|\vec{\mathbf{r}}'(t)|} \right\rangle.$$

**Theorem 263** (Line Integral of Normal Component)**.** Let $C$ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let $D$ be the region bounded by $C$. Let $\vec{\mathbf{n}}$ denote the outward unit normal vector of $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then the line integral of the normal component of $\vec{\mathbf{F}}$ along a curve $C$ is

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = \iint_D \text{div } \vec{\mathbf{F}}(x, y)\, dA.$$

Suggested Problems: p. 1097, 1-8, 13-18,

# 16.6 Parametric Surfaces and Their Areas

## 16.6.1 Parametric Surfaces

**Goal 264.** This section will aim to describe surfaces by a function $\vec{\mathbf{r}}(u,v) = x(u,v)\widehat{\mathbf{i}} + y(u,v)\widehat{\mathbf{j}} + z(u,v)\widehat{\mathbf{k}}$ in a similar way that we described vector functions by $\vec{\mathbf{r}}(t)$ in previous chapters.

**Problem 265.** Identify and sketch the surface with vector equation $\vec{\mathbf{r}}(u,v) = \langle 2\cos(u), v, 2\sin(u)\rangle$ .

**Problem 266.** Find a parametric representation of the sphere $x^2 + y^2 + z^2 = a^2$.

**Problem 267.** Find a parametric representation for the cylinder $x^2 + y^2 = 4$, where $0 \leq z \leq 1$.

**Problem 268.** Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

**Problem 269.** Find a parametric representation for the surface $z = 2\sqrt{x^2 + y^2}$, that is, the top half of the cone $z^2 = 4x^2 + 4y^2$.

### 16.6.2 Surface of Revolution

**Theorem 270** (Parametrization of Surface of Revolution)**.** Consider a surface $S$ that can be obtained by rotating a curve $y = f(x)$ from $a \leq x \leq b$ about the $x$-axis, where $f(x) \geq 0$. Let $\theta$ be the angle of ration of the surface. Then the surface can be parametrized by

$$x = x \qquad y = f(x)\cos(\theta) \qquad z = f(x)\sin(\theta).$$

**Problem 271.** Find parametric equations for the surface generated by rotating the curve $y = \sin(x)$ from $0 \leq x \leq 2\pi$ about the $x$-axis.

### 16.6.3 Tangent Planes

**Theorem 272** (Normal Vector to a Surface)**.** Let $S$ be a surface defined by $\vec{\mathbf{r}}(u, v) = x(u, v)\widehat{\mathbf{i}} + y(u, v)\widehat{\mathbf{j}} + z(u, v)\widehat{\mathbf{k}}$. To define the tangent plane at a point $P_0 = (u_0, v_0)$, we must first find a normal vector to the surface at $P_0$. In this vein, define

$$\begin{aligned} \vec{\mathbf{r}}_v &= \left\langle \tfrac{\partial x}{\partial v}(u_0, v_0), \tfrac{\partial y}{\partial v}(u_0, v_0), \tfrac{\partial z}{\partial v}(u_0, v_0) \right\rangle \\ \vec{\mathbf{r}}_u &= \left\langle \tfrac{\partial x}{\partial u}(u_0, v_0), \tfrac{\partial y}{\partial u}(u_0, v_0), \tfrac{\partial z}{\partial u}(u_0, v_0) \right\rangle . \end{aligned}$$

We use these to give us a normal vector $\vec{\mathbf{n}}(u_0, v_0) = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$. This can be used to find the equation of a tangent plane as in 12.2

**Problem 273.** Find the tangent plane to the surface with parametric equations $x = u^2, y = v^2$, and $z = u + 2v$ at the point $(1, 1, 3)$.

### 16.6.4 Surface Area

**Definition 274** (Surface Area)**.** If a smooth parametric surface $S$ is given by the equation $\vec{\mathbf{r}}(u,v) = x(u,v)\widehat{\mathbf{i}} + y(u,v)\widehat{\mathbf{j}} + z(u,v)\widehat{\mathbf{k}}$, $(u,v) \in D$ and $S$ is covered just once as $(u,v)$ ranges through the parameter domain $D$, then the **surface area** of $S$ is

$$A(S) = \iint\limits_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| \, dA,$$

where $\vec{\mathbf{r}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$ and $\vec{\mathbf{r}}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$.

**Problem 275.** Find the surface area of a sphere with radius $a$.

Suggested Problems: p. $1108, 3-6, 19-26, 37, 38, 39-45$.

## 16.7 Surface Integrals

**Definition 276** (Surface Integral)**.** Suppose $S$ is a surface with vector equation $\vec{\mathbf{r}}(u,v) = x(u,v)\widehat{\mathbf{i}} + y(u,v)\widehat{\mathbf{j}} + z(u,v)\widehat{\mathbf{k}}$, $(u,v) \in D$. Then the **surface integral of $f$ over the surface** $S$ is

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f\left(\vec{\mathbf{r}}\,(u,v)\right)\left|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right|\,dA.$$

where $\vec{\mathbf{r}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$ and $\vec{\mathbf{r}}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$.

**Problem 277.** Compute the surface integral $\iint\limits_S x^2\,dS$, where $S$ is the unit sphere $x^2 + y^2 + z^2 = 1$.

**Theorem 278** (Surface Integral of Graphs)**.** Let $S$ be a surface with equation $z = g(x,y)$. Moreover, $S$ has parametrization

$$x = x \qquad y = y \qquad z = g(x,y),$$

and the **surface integral** is

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f\left(x,y,g\left(x,y\right)\right)\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA.$$

**Problem 279.** Evaluate $\iint\limits_S y\,dS$, where $S$ is the surface $z = x + y^2$ from $0 \le x \le 1$ and $0 \le y \le 2$.

**Definition 280** (Orientable Surface)**.** A surface is called **orientable** if it has two sides.

**Definition 281** (Surface Integral over Vector Field)**.** If $\vec{\mathbf{F}}$ is a continuous vector field defined on an oriented surface $S$ with unit normal vector $\vec{\mathbf{n}}$, then the **surface integral of $\vec{\mathbf{F}}$ over** $S$ is

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS = \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA.$$

This integral is also called the **flux** of $\vec{\mathbf{F}}$ across $S$.

**Problem 282.** Find the flux of the vector field $\vec{\mathbf{F}}(x, y, z) = \langle z, y, x\rangle$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

**Problem 283.** Evaluate $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$, where $\vec{\mathbf{F}}(x, y, z) = \langle y, x, z\rangle$ and $S$ is the boundary of the solid region $E$ enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Suggested Problems: p. $1120, 5 - 10, 21 - 25$

## 16.8 Stokes' Theorem

**Theorem 284** (Stoke's Theorem)**.** Let $S$ be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve $C$ with positive orientation. Let $\vec{\mathbf{F}}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains $S$. Then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \text{curl } \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}.$$

**Problem 285.** Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y, z) = \langle -y^2, x, z^2 \rangle$ and $C$ is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

**Problem 286.** Use Stoke's Theorem to compute the integral $\iint_S \text{curl } \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$, where $\vec{\mathbf{F}}(x, y, z) = \langle xx, yz, xy \rangle$ and $S$ is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the $xy$-plane.

Suggested Problems: p. $1127, 2 - 10$.

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